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How to construct indecomposable entanglement witnesses

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Abstract
We present a very simple method for constructing indecomposable entanglement witnesses out of a given pair—an entanglement witness W and the corresponding state detected by W. This method may be used to produce new classes of atomic witnesses which are able to detect the 'weakest' quantum entanglement. Actually, it works perfectly in the multipartite case, too. Moreover, this method provides a powerful tool for constructing new examples of bound entangled states.

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1. Introduction

One of the most important problems of quantum information theory [1, 2] is the characterization of mixed states of composed quantum systems. In particular it is of primary importance testing whether a given quantum state is separable or entangled. For low dimensional systems there exists simple necessary and sufficient condition for separability. The celebrated Peres–Horodecki criterion [3, 4] states that a state of a bipartite system living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable if its partial transpose is positive, i.e. the state is PPT. Unfortunately, for higher-dimensional systems there is no single universal separability condition.

The most general approach to characterize quantum entanglement uses a notion of an entanglement witness (EW) [5–8]. A Hermitian operator W defined on a tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is called an EW iff (1) $\mathrm{Tr}(W\sigma_{\rm sep}) \geqslant 0$ for all separable states $\sigma_{\rm sep}$, and (2) there exists an entangled state ρ such that $\mathrm{Tr}(W\rho) < 0$ (one says that ρ is detected by W). It turns out that a state is entangled if and only if it is detected by some EW [5]. There have been considerable efforts in constructing and analyzing the structure of EWs [6–15]. In fact, entanglement witnesses have already been measured in several experiments [16, 17]. Moreover, several procedures for optimizing EWs for arbitrary states were proposed [8, 18–20].

The simplest way to construct an EW is to define $W = P + (\mathbb{1} \otimes \tau)Q$, where P and Q are positive operators, and $(\mathbb{1} \otimes \tau)Q$ denotes partial transposition. It is easy to see that

$\mathrm{Tr}(W\sigma_{\mathrm{sep}}) \geqslant 0$ for all separable states σ_{sep}, and hence if W is non-positive, then it is an EW. Such EWs are said to be decomposable [8]. Note, however, that decomposable EWs cannot detect PPT entangled state (PPTES) and, therefore, such EWs are useless in search of bound entangled state. Unfortunately, there is no general method to construct indecomposable EW and only very few examples of indecomposable EWs are available in the literature. For example the authors of [9] proposed very interesting method for constructing EWs based on the knowledge of edge states. This construction works very nice provided one knows this particular class of states. However, the general construction of edge states is not known and hence the method of [9], although interesting and important from theoretical point of view, is not effective.

In the present paper we propose a very simple method for constructing indecomposable EWs. If we are given one indecomposable EW W_0 and the corresponding state ρ_0 detected by W_0, then we are able to construct an open convex set of indecomposable EWs detecting ρ_0, and an open convex set of PPTES detected by W_0. Hence, out of a given pair (W_0, ρ_0) we construct huge classes of new EWs and PPTES, respectively. In particular, we may apply this method to construct so called atomic EWs which are able to detect the 'weakest' quantum entanglement (i.e. PPTES ρ such that both Schmidt number [21] of ρ and its partial transposition $(\mathbb{1} \otimes \tau)\rho$ does not exceed 2). We stress that proposed method is very general and works perfectly for multipartite case.

The paper is organized as follows: in the next section we introduce a natural hierarchy of convex cones in space of EWs. This hierarchy explains importance of indecomposable and atomic EWs. Section 3 presents our method for constructing indecomposable EWs. Section 4 provides construction of atomic EWs which is illustrated by a new class of such witnesses. Finally, in section 5 we generalize our construction for multipartite case. A brief discussion is included in the last section.

2. A hierarchy of entanglement witnesses

Consider a space $\mathcal{P}$ of positive operators in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Let us recall [21] that for any normalized positive operator σ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ one may define its Schmidt number

$$\mathrm{SN}(\sigma) = \min_{p_k,\psi_k}\{\max_k \mathrm{SR}(\psi_k)\}, \tag{1}$$

where the minimum is taken over all possible pure states decompositions

$$\sigma = \sum_k p_k|\psi_k\rangle\langle\psi_k|, \tag{2}$$

with $p_k \geqslant 0$, $\sum_k p_k = 1$ and ψ_k are normalized vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$. The Schmidt rank $\mathrm{SR}(\psi)$ denotes the number of non-vanishing Schmidt coefficients in the Schmidt decomposition of ψ. This number characterizes the minimum Schmidt rank of pure states that are needed to construct such density matrix. It is evident that $1 \leqslant \mathrm{SN}(\rho) \leqslant d$, with $d = \min\{d_1, d_2\}$, and $d_k = \dim\mathcal{H}_k$. Moreover, ρ is separable iff $\mathrm{SN}(\rho) = 1$. It was proved [21] that the Schmidt number is non-increasing under local operations and classical communication.

Now, the notion of the Schmidt number enables one to introduce a natural family of convex cones in $\mathcal{P}$:

$$\mathbf{V}_r = \{\rho \in \mathcal{P} | \mathrm{SN}(\rho) \leqslant r\}. \tag{3}$$

One has the following chain of inclusions

$$\mathbf{V}_1 \subset \ldots \subset \mathbf{V}_d = \mathcal{P}, \tag{4}$$

where $d = \min\{d_1, d_2\}$, and $d_k = \dim\mathcal{H}_k$. Clearly, $\mathbf{V}_1$ is a cone of separable (unnormalized) states and $\mathbf{V}_d \setminus \mathbf{V}_1$ stands for a set of entangled states. Note, that a partial transposition $(\mathbb{1} \otimes \tau)$ gives rise to another family of cones:

$$\mathbf{V}^l = (\mathbb{1} \otimes \tau)\mathbf{V}_l, \tag{5}$$

such that $\mathbf{V}^1 \subset \ldots \subset \mathbf{V}^d$. One has $\mathbf{V}_1 = \mathbf{V}^1$, together with the following hierarchy of inclusions:

$$\mathbf{V}_1 = \mathbf{V}_1 \cap \mathbf{V}^1 \subset \mathbf{V}_2 \cap \mathbf{V}^2 \subset \ldots \subset \mathbf{V}_d \cap \mathbf{V}^d. \tag{6}$$

Note, that $\mathbf{V}_d \cap \mathbf{V}^d$ is a convex set of PPT (unnormalized) states. Finally, $\mathbf{V}_r \cap \mathbf{V}^s$ is a convex subset of PPT states ρ such that $\mathrm{SN}(\rho) \leqslant r$ and $\mathrm{SN}[(\mathbb{1} \otimes \tau)\rho] \leqslant s$.

Now, in the set of entanglement witnesses $\mathbf{W}$ one may introduce the family of dual cones:

$$\mathbf{W}_r = \{W \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \,|\, \mathrm{Tr}(W\rho) \geqslant 0, \rho \in \mathbf{V}_r\}. \tag{7}$$

One has

$$\mathcal{P} = \mathbf{W}_d \subset \ldots \subset \mathbf{W}_1. \tag{8}$$

Clearly, $\mathbf{W} = \mathbf{W}_1 \setminus \mathbf{W}_d$. Moreover, for any $k > l$, entanglement witnesses from $\mathbf{W}_l \setminus \mathbf{W}_k$ can detect entangled states from $\mathbf{V}_k \setminus \mathbf{V}_l$, i.e. states ρ with Schmidt number $l < \mathrm{SN}(\rho) \leqslant k$. In particular $W \in \mathbf{W}_k \setminus \mathbf{W}_{k+1}$ can detect state ρ with $\mathrm{SN}(\rho) = k$.

Finally, let us consider the following class

$$\mathbf{W}_r^s = \mathbf{W}_r + (\mathbb{1} \otimes \tau)\mathbf{W}_s, \tag{9}$$

that is, $W \in \mathbf{W}_r^s$ iff

$$W = P + (\mathbb{1} \otimes \tau)Q, \tag{10}$$

with $P \in \mathbf{W}_r$ and $Q \in \mathbf{W}_s$. Note, that $\mathrm{Tr}(W\rho) \geqslant 0$ for all $\rho \in \mathbf{V}_r \cap \mathbf{V}^s$. Hence such W can detect PPT states ρ such that $\mathrm{SN}(\rho) \geqslant r$ and $\mathrm{SN}[(\mathbb{1} \otimes \tau)\rho] \geqslant s$. Entanglement witnesses from $\mathbf{W}_d^d$ are called decomposable [8]. They cannot detect PPT states. One has the following chain of inclusions:

$$\mathbf{W}_d^d \subset \ldots \subset \mathbf{W}_2^2 \subset \mathbf{W}_1^1 \equiv \mathbf{W}. \tag{11}$$

To deal with PPT states one needs indecomposable witnesses from $\mathbf{W}^{\mathrm{ind}} := \mathbf{W} \setminus \mathbf{W}_d^d$. The 'weakest' entanglement can be detected by elements from $\mathbf{W}^{\mathrm{atom}} := \mathbf{W} \setminus \mathbf{W}_2^2$. We shall call them *atomic entanglement witnesses*. It is clear that W is an atomic entanglement witness if there is an entangled state $\rho \in \mathbf{V}_2 \cap \mathbf{V}^2$ such that$\mathrm{Tr}(W\rho) < 0$. The knowledge of atomic witnesses, or equivalently atomic maps, is crucial: knowing this set would enable us to distinguish all entangled states from separable ones.

3. Detecting PPT entangled states

Suppose that a PPT entangled state ρ_0 in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is detected by an entanglement witness W_0, that is

$$\mathrm{Tr}(W_0\rho_0) < 0. \tag{12}$$

It is clear that in the vicinity of ρ_0 there are other PPT entangled states detected by the same witness W_0. Let σ_{sep} be an arbitrary separable state and consider the following convex combination

$$\rho_\alpha = (1 - \alpha)\rho_0 + \alpha\sigma_{\mathrm{sep}}. \tag{13}$$

It is evident that ρ_α is PPT for any $\alpha \in [0, 1]$. Moreover, for any $0 \leqslant \alpha < \alpha_{[\rho_0,\sigma_{\rm sep}]}$, with

$$\alpha_{[\rho_0,\sigma_{\rm sep}]} := \sup\{\alpha \in [0,1] | \operatorname{Tr}(W_0\rho_\alpha) < 0\}, \tag{14}$$

ρ_α is entangled. This construction gives rise to an open convex set

$$\mathcal{S}^{\rm PPT}[W_0|\rho_0] := \big\{\rho_\alpha \big| 0 \leqslant \alpha < \alpha_{[\rho_0,\sigma_{\rm sep}]} \text{ \& aribitrary } \sigma_{\rm sep}\big\}. \tag{15}$$

All elements from $\mathcal{S}^{\rm PPT}[W_0|\rho_0]$ are PPT entangled states detected by W_0. On the other hand in the vicinity of W_0 there are other entanglement witnesses detecting our original PPT state ρ_0. Indeed, let P be an arbitrary positive semidefinite operator in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and consider one-parameter family of operators

$$W_\lambda = W_0 + \lambda P, \lambda \geqslant 0. \tag{16}$$

Let us observe that for any $0 \leqslant \lambda < \lambda_{[W_0,P]}$ with

$$\lambda_{[W_0,P]} := \sup\{\lambda \geqslant 0 | \operatorname{Tr}(W_\lambda\rho_0) < 0\}, \tag{17}$$

W_λ is an indecomposable EW detecting a PPT state ρ_0. This construction gives rise to a dual open convex set

$$\mathcal{W}^{\rm ind}[W_0|\rho_0] := \big\{W_\lambda \big| 0 \leqslant \lambda < \lambda_{[W_0,P]} \text{ \& aribitrary } P \geqslant 0\big\}. \tag{18}$$

Summarizing, having a pair of a PPTES ρ_0 and an indecomposable EW W_0 we may construct two open convex sets: $\mathcal{S}^{\rm PPT}[W_0|\rho_0]$ containing PPTES detected by W_0 and $\mathcal{W}^{\rm ind}[W_0|\rho_0]$ containing an indecomposable EW detecting ρ_0. It shows that for any $\rho_1, \rho_2 \in \mathcal{S}^{\rm PPT}[W_0|\rho_0]$ any convex combination

$$p_1\rho_1 + p_2\rho_2 \in \mathcal{S}^{\rm PPT}[W_0|\rho_0], \tag{19}$$

and hence defines a PPTES. Similarly, for any $W_1, W_2 \in \mathcal{W}^{\rm ind}[W_0|\rho_0]$ any convex combination

$$w_1W_1 + w_2W_2 \in \mathcal{W}^{\rm ind}[W_0|\rho_0], \tag{20}$$

and hence defines an indecomposable EW. Therefore, the above constructions provide a method to produce new PPTES and new indecomposable EW out of a single pair (ρ_0, W_0).

Note, that this construction may easily be continued. Let us take an arbitrary EW W' from $\mathcal{W}^{\rm ind}[W_0|\rho_0]$ (different from W_0). It is easy to find PPTES from $\mathcal{S}^{\rm PPT}[W_0|\rho_0]$ detected by W': indeed, any state in $\mathcal{S}^{\rm PPT}[W_0|\rho_0]$ has a form (13) and hence

$$\operatorname{Tr}(W'\rho_\alpha) = (1-\alpha)\operatorname{Tr}(W'\rho_0) + \alpha\operatorname{Tr}(W'\sigma_{\rm sep}). \tag{21}$$

Therefore, one has $\operatorname{Tr}(W'\rho_\alpha) < 0$ for

$$\alpha < \frac{-\operatorname{Tr}(W'\rho_0)}{-\operatorname{Tr}(W'\rho_0) + \operatorname{Tr}(W'\sigma_{\rm sep})} \leqslant 1. \tag{22}$$

Now, W' and $\rho' = \rho_\alpha$ with α satisfying (22) define a new pair which may be used as a starting point for the construction of $\mathcal{S}^{\rm PPT}[W'|\rho']$ and $\mathcal{W}^{\rm ind}[W'|\rho']$.

4. Constructing atomic entanglement witnesses

Suppose now, that we are given a 'weakly entangled' PPTES, i.e. a state $\rho_0 \in \mathbf{V}_2 \cap \mathbf{V}^2$ and let W_0 be the corresponding atomic EW. Following our construction we define

$$\mathcal{S}_2^2[W_0|\rho_0] \subset \mathbf{V}_2 \cap \mathbf{V}^2, \tag{23}$$

such that each element from $\mathcal{S}_2^2[W_0|\rho_0]$ is detected by the same witness W_0. Similarly, we define a set of atomic witnesses

$$\mathcal{W}^{\rm atom}[W_0|\rho_0] \subset \mathbf{W}^{\rm atom}, \tag{24}$$

such that each element from $\mathcal{W}^{\text{atom}}[W_0|\rho_0]$ detects our original state ρ_0. Both sets $\mathcal{S}_2^2[W_0|\rho_0]$ and $\mathcal{W}^{\text{atom}}[W_0|\rho_0]$ are open and convex.

Note, that knowing atomic EWs one may detect all entangled states. Moreover, it was conjectured by Osaka in [22] that all EWs in $\mathcal{B}(\mathbb{C}^3 \otimes \mathbb{C}^3)$ can be represented as a sum of decomposable and atomic witnesses. To best of our knowledge this conjecture is still open. It shows that the knowledge of atomic EWs is crucial both from physical and purely mathematical points of view.

It is well known that there is a direct relation between entanglement witnesses in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and linear positive maps $\varphi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$, i.e. maps which send positive elements from$\mathcal{B}(\mathcal{H}_1)$ into positive elements from $\mathcal{B}(\mathcal{H}_2)$.[1] One calls a linear positive map φk-positive iff the following map

$$\varphi^{(k)} := \text{id}_k \otimes \varphi : \mathcal{B}(\mathbb{C}^k \otimes \mathcal{H}_1) \longrightarrow \mathcal{B}(\mathbb{C}^k \otimes \mathcal{H}_2), \tag{25}$$

is positive ('id_k' denotes an identity map in the matrix algebra $M_k = \mathcal{B}(\mathbb{C}^k)$). If $\varphi^{(k)}$ is positive for all integers $k = 1, 2, \ldots$, then one calls the original map φ completely positive.

Now, to describe relations between positive maps and Hermitian operators in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ let us introduce the following notation: $(e_1, \ldots, e_d)$ denotes an orthonormal basis in $\mathbb{C}^d$, and $e_{ij} = |e_i\rangle\langle e_j|$. Note, that the canonical maximally entangled state in $\mathcal{H}_1 \otimes \mathcal{H}_1$

$$\psi^+_{d_1} := \frac{1}{\sqrt{d_1}} \sum_{i=1}^{d_1} e_i \otimes e_i, \tag{26}$$

gives rise to the following operator in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$

$$W_\varphi = (\text{id}_{d_1} \otimes \varphi) P^+_{d_1}, \tag{27}$$

where $P^+_{d_1} = d_1 \big|\psi^+_{d_1}\big\rangle\big\langle\psi^+_{d_1}\big|$. One has therefore the following relation:

$$\varphi \longrightarrow W_\varphi := \sum_{i,j=1}^{d_1} e_{ij} \otimes \varphi(e_{ij}), \tag{28}$$

known as a Choi–Jamiołkowski isomorphism [23, 24]. It is shown in [23, 24] that φ is a positive map if and only if $\text{Tr}(W_\varphi \sigma) \geqslant 0$ for any separable state σ. Moreover, φ is completely positive if and only if W_φ defines a positive operator, i.e. $\text{Tr}(W_\varphi \rho) \geqslant 0$ for any (not necessarily separable) state ρ. Summarizing, any positive but not completely positive map $\varphi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$ gives rise to an EW W_φ. If φ is indecomposable, i.e. it can not be written as a sum $\phi_1 + \phi_2 \circ \tau$, where ϕ_1 and ϕ_2 are completely positive, then W_φ defines an indecomposable EW. If, moreover, φ is atomic, i.e. can not be written as a sum $\phi_1 + \phi_2 \circ \tau$, where ϕ_1 and ϕ_2 are 2-positive, then W_φ defines an atomic EW.

Now, we are illustrate how our method works in practice.

4.1. Generalizing a Choi EW in $3 \otimes 3$

Let us recall the celebrated positive map $\varphi : M_3 \longrightarrow M_3$ introduced by Choi [24]:

$$\varphi(e_{11}) = e_{11} + e_{22}, \qquad \varphi(e_{22}) = e_{22} + e_{33}, \qquad \varphi(e_{33}) = e_{33} + e_{11}, \tag{29}$$

[1] Recall, that $\mathcal{B}(\mathcal{H}_1) \ni A \geqslant 0$ if $A = B^* B$ for some elements $B \in \mathcal{B}(\mathcal{H}_1)$.

and $\varphi(e_{ij}) = -e_{ij}$, for $i \neq j$. Consider the corresponding operator in $M_3 \otimes M_3$ which is related via Choi–Jamiołkowski isomorphism to the Choi map. One easily finds

$$W_0 = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 \end{array}\right), \tag{30}$$

where maintaining a more transparent form we replace all zeros by dots. It was shown by Ha [25] that W_0 is atomic. The proof is based on construction of a state in $\mathbf{V}_2 \cap \mathbf{V}^2$ detected by W_0. Actually, Ha constructed a whole one-parameter family of such states. For any $0 < \gamma < 1$ let us define

$$\rho_\gamma = \frac{1}{N_\gamma} \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & a_\gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b_\gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & b_\gamma & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_\gamma & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_\gamma & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_\gamma & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{array}\right), \tag{31}$$

with

$$a_\gamma = \tfrac{1}{3}(\gamma^2 + 2), \qquad b_\gamma = \tfrac{1}{3}(\gamma^{-2} + 2), \tag{32}$$

and the normalization factor

$$N_\gamma = 7 + \gamma^2 + \gamma^{-2}. \tag{33}$$

It was shown [25] that $\rho_\gamma \in \mathbf{V}_2 \cap \mathbf{V}^2$ and $\mathrm{Tr}(W_0 \rho_\gamma) = (\gamma^2 - 1)/N_\gamma$. Hence, for $\gamma < 1$ the state ρ_γ is entangled (and W_0 is an indecomposable EW).[2] It is therefore clear that if $\gamma_1, \ldots, \gamma_K \in (0, 1)$, then any convex combination

$$p_1 \rho_{\gamma_1} + \ldots + p_K \rho_{\gamma_K} \tag{35}$$

defines an entangled state in $\mathbf{V}_2 \cap \mathbf{V}^2$ detected by W_0.

Consider now the following maximally entangled state in $\mathbb{C}^3 \otimes \mathbb{C}^3$:

$$\psi = \tfrac{1}{\sqrt{3}}(e_1 \otimes e_3 + e_2 \otimes e_1 + e_3 \otimes e_2), \tag{36}$$

[2] Actually, for $\gamma = 1$ one has

$$\rho_{\gamma=1} = \frac{1}{9} \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{array}\right), \tag{34}$$

and it is known [4] that this state is separable.

and let $P = 3|\psi\rangle\langle\psi|$. Define $W_\lambda = W_0 + \lambda P$. It is given by the following matrix

$$W_\lambda = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda & \lambda & \cdot & \cdot & \cdot & \lambda & \cdot \\ \hline \cdot & \cdot & \lambda & \lambda & \cdot & \cdot & \cdot & \lambda & \cdot \\ -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \lambda & \lambda & \cdot & \cdot & \cdot & \lambda & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 \end{array}\right), \tag{37}$$

and hence $\mathrm{Tr}(W_\lambda \rho_\gamma) < 0$, if

$$\lambda < \frac{1-\gamma^2}{2+\gamma^{-2}}. \tag{38}$$

Actually, the maximal value of λ is attainable for $\gamma^* = \sqrt{(\sqrt{3}-1)/2} \approx 0.605$. Therefore, taking as ρ_0 the state ρ_{γ^*}, one finds $\lambda_{[W_0,P]} = (1-\gamma^{*2})/(2+\gamma^{*-2}) \approx 0.133$. This way it is shown that W_λ, with $0 \leqslant \lambda < \lambda_{[W_0,P]}$, defines an atomic EW. We may still modify W_λ by adding for example a positive operator $Q = 3|\varphi\rangle\langle\varphi|$, where

$$\varphi = \tfrac{1}{\sqrt{3}}(e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes e_1), \tag{39}$$

that is

$$W_{\lambda,\mu} = W_0 + \lambda P + \mu Q. \tag{40}$$

One finds the following matrix representation

$$W_{\lambda,\mu} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 1+\mu & \cdot & \cdot & \cdot & \mu & \mu & \cdot & \cdot \\ \cdot & \cdot & \lambda & \lambda & \cdot & \cdot & \cdot & \lambda & \cdot \\ \hline \cdot & \cdot & \lambda & \lambda & \cdot & \cdot & \cdot & \lambda & \cdot \\ -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \mu & \cdot & \cdot & \cdot & 1+\mu & \mu & \cdot & \cdot \\ \hline \cdot & \mu & \cdot & \cdot & \cdot & \mu & 1+\mu & \cdot & \cdot \\ \cdot & \cdot & \lambda & \lambda & \cdot & \cdot & \cdot & \lambda & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 \end{array}\right). \tag{41}$$

Now, $\mathrm{Tr}(W_{\lambda,\mu}\rho_\gamma) < 0$, if λ satisfies (38) and

$$\mu < \frac{1-\gamma^2-\lambda(2+\gamma^{-2})}{2+\gamma^2}. \tag{42}$$

Interestingly, applying our method to a pair (W_0, ρ_γ) we constructed a whole class of atomic EWs $W_{\lambda,\mu}$ which have a circulant structure analyzed in [26]. Therefore, it may be used testing quantum entanglement within a class of circulant PPT states [26] (see also [27]). To best of our knowledge this is the first example of a 'circulant atomic' EW. Consider for example the following (unnormalized) state [26]

$$\rho = \left(\begin{array}{ccc|ccc|ccc} a_{11} & \cdot & \cdot & \cdot & a_{12} & \cdot & \cdot & \cdot & a_{13} \\ \cdot & b_{11} & \cdot & \cdot & \cdot & b_{12} & b_{13} & \cdot & \cdot \\ \cdot & \cdot & c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{13} & \cdot \\ \hline \cdot & \cdot & c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{23} & \cdot \\ a_{21} & \cdot & \cdot & \cdot & a_{22} & \cdot & \cdot & \cdot & a_{23} \\ \cdot & b_{21} & \cdot & \cdot & \cdot & b_{22} & b_{23} & \cdot & \cdot \\ \hline \cdot & b_{31} & \cdot & \cdot & \cdot & b_{32} & b_{33} & \cdot & \cdot \\ \cdot & \cdot & c_{31} & c_{32} & \cdot & \cdot & \cdot & c_{33} & \cdot \\ a_{31} & \cdot & \cdot & \cdot & a_{32} & \cdot & \cdot & \cdot & a_{33} \end{array}\right), \tag{43}$$

where $a = [a_{ij}]$, $b = [b_{ij}]$ and $c = [c_{ij}]$ are 3×3 positive matrices. One easily finds that the condition $\mathrm{Tr}(\rho W_{\lambda,\mu}) < 0$ leads to the following conclusion: if

$$\mathrm{Tr}(2a + b + c) + \mathrm{Tr}(J[\mu b + \lambda c]) < \mathrm{Tr}(Ja), \tag{44}$$

where $J = [J_{ij}]$ is a 3×3 matrix with $J_{ij} = 1$, then ρ is an entangled PPT state.

4.2. Atomic EW in $d \otimes d$

Actually, the example analyzed in the previous section may be generalized for $d \otimes d$ case. Consider the following set of Hermitian operators:

$$W_{d,k} := \sum_{i,j=1}^{d} e_{ij} \otimes X_{ij}^{d,k}, \tag{45}$$

where $d \times d$ matrices $X_{ij}^{d,k}$ are defined as follows:

$$X_{ij}^{d,k} = \begin{cases} (d-k-1)e_{ii} + \displaystyle\sum_{l=1}^{k} e_{i+l,i+l}, & i = j \\ -e_{ij}, & i \neq j. \end{cases} \tag{46}$$

For $d = 3$ and $k = 1$ the above formula reconstructs W_0 defined in (30). Again, $W_{d,k}$ are related via Choi–Jamiołkowski isomorphism to the family of positive maps [28]

$$\tau_{d,k}(x) = (d-k)\varepsilon(x) + \sum_{l=1}^{k} \varepsilon(S^l x S^{*l}) - x, x \in M_d, \tag{47}$$

where $\varepsilon(x) = \sum_{i=1}^{d} x_{ii} e_{ii}$, and S is the shift operator defined by $Se_i = e_{i+1} (\mathrm{mod}\, d)$. The positivity of $\tau_{d,k}$ for $k = 1, \ldots, d-1$ was shown in [28] (for $k = d-1$ this map is completely copositive) and Osaka showed that $\tau_{d,1}$ is atomic. Finally, it was shown by Ha in [25] that it is atomic for $k = 1, \ldots, k-2$. Therefore, it proves the atomicity of $W_{d,k}$. Ha's proof is based on the construction of the family of states $\rho_\gamma \in \mathbf{V}_2 \cap \mathbf{V}^2$:

$$\rho_\gamma = \frac{1}{N_\gamma} \sum_{i,j=1}^{d} e_{ij} \otimes A_{ij}^{\gamma}, \tag{48}$$

where $d \times d$ matrices A_{ij}^{γ} are defined as follows:

$$A_{ij}^{\gamma} = \begin{cases} e_{ij}, & i \neq j \\ e_{11} + a_\gamma e_{22} + \displaystyle\sum_{l=3}^{d-1} e_{ll} + b_\gamma e_{dd}, & i = j = 1 \\ S^{j-1} A_{11} S^{*j-1}, & i = j \neq 1, \end{cases} \tag{49}$$

with

$$a_\gamma = \frac{1}{d}(\gamma^2 + d - 1), \qquad b_\gamma = \frac{1}{d}(\gamma^{-2} + d - 1), \tag{50}$$

and the normalization factor

$$N_\gamma = d^2 - 2 + \gamma^2 + \gamma^{-2}, \tag{51}$$

which reproduces (33) for $d = 3$. One shows [25] that $\rho_\gamma \in \mathbf{V}_2 \cap \mathbf{V}^2$ and $\mathrm{Tr}(W_{d,k}\rho_\gamma) = (\gamma^2 - 1)/N_\gamma$. Hence, for $\gamma < 1$, the family of states ρ_γ is detected by each $W_{d,k}$ for $k = 1, \ldots, d-2$. It is therefore clear that any convex combination

$$W_d[\mathbf{p}] := \sum_{k=1}^{d-2} p_k W_{d,k}, \qquad \mathbf{p} = (p_1, \ldots, p_{d-2}), \tag{52}$$

gives rise to a new atomic EW $W_d[\mathbf{p}]$. Following three-dimensional example one may easily construct out of a pair $(W_{d,k}, \rho_\gamma)$ a family of new EWs.

5. Multipartite entanglement witnesses

Let us note, that the above construction works perfectly for multipartite case. Consider N-partite system living in $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$. A state ρ_0 in $\mathcal{H}$ is entangled if there exists an entanglement witness $W_0 \in \mathcal{B}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N)$ such that:

(1) $\mathrm{Tr}(W_0\sigma_{\mathrm{sep}}) \geqslant 0$ for all N-separable states σ_{sep},
(2) $\mathrm{Tr}(W_0\rho_0) < 0$.

In the multipartite case a set of PPT states may be generalized as follows. For each binary N-vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ one introduces a class of $\boldsymbol{\sigma}$-PPT states: ρ is $\boldsymbol{\sigma}$-PPT iff

$$\tau^{\boldsymbol{\sigma}}\rho := (\tau^{\sigma_1} \otimes \ldots \otimes \tau^{\sigma_N})\rho \geqslant 0. \tag{53}$$

Finally, an entanglement witness W is $\boldsymbol{\sigma}$-decomposable if it can be represented as the following sum

$$W = Q_1 + \tau^{\boldsymbol{\sigma}} Q_2, \tag{54}$$

where Q_1 and Q_2 are positive operators in $\mathcal{B}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N)$. Clearly, $\boldsymbol{\sigma}$-decomposable EW cannot detect entangled $\boldsymbol{\sigma}$-PPT state.

Suppose, that an entangled N-partite $\boldsymbol{\sigma}$-PPT state ρ_0 is detected by $\boldsymbol{\sigma}$-indecomposable entanglement witness W_0. Therefore, if σ_{sep} is an arbitrary N-separable state, then the following convex combination

$$\rho_\alpha = (1-\alpha)\rho_0 + \alpha\sigma_{\mathrm{sep}}, \tag{55}$$

defines $\boldsymbol{\sigma}$-PPT entanglement state for any $0 \leqslant \alpha < \alpha_{[\rho_0,\sigma_{\mathrm{sep}}]}$, with

$$\alpha_{[\rho_0,\sigma_{\mathrm{sep}}]} := \sup\{\alpha \in [0,1] \,|\, \mathrm{Tr}(W_0\rho_\alpha) < 0\}. \tag{56}$$

This construction gives rise to an open convex set

$$\mathcal{S}^{\mathrm{PPT}}_{\boldsymbol{\sigma}}[W_0|\rho_0] := \left\{\rho_\alpha \middle| 0 \leqslant \alpha < \alpha_{[\rho_0,\sigma_{\mathrm{sep}}]} \ \&\ \text{aribitrary } \sigma_{\mathrm{sep}}\right\}. \tag{57}$$

Similarly, let P be an arbitrary positive semidefinite operator in $\mathcal{B}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N)$ and consider one-parameter family of operators

$$W_\lambda = W_0 + \lambda P, \qquad \lambda \geqslant 0. \tag{58}$$

Let us observe that for any $0 \leqslant \lambda < \lambda_{[W_0,P]}$ with

$$\lambda_{[W_0,P]} := \sup\{\lambda \geqslant 0 \,|\, \mathrm{Tr}(W_\lambda\rho_0) < 0\}, \tag{59}$$

W_λ defines $\boldsymbol{\sigma}$-indecomposable EW detecting the state ρ_0. This construction gives rise to a dual open convex set

$$\mathcal{W}^{\mathrm{ind}}_{\boldsymbol{\sigma}}[W_0|\rho_0] := \left\{W_\lambda \middle| 0 \leqslant \lambda < \lambda_{[W_0,P]} \ \&\ \text{aribitrary } P \geqslant 0\right\}. \tag{60}$$

6. Conclusions

A simple and general method for constructing indecomposable EWs was presented. Knowing an EW W_0 and the corresponding entangled PPT state ρ_0 detected by W_0, one should be able to construct new EWs and new PPTES. In particular one may apply this method to construct new examples of atomic EWs which are crucial when distinguishing between separable and entangled states. Moreover, one may apply the same strategy in constructing EWs for multipartite systems also.

What can we do if only one element from the above pair is available? Note, that a nonpositive Hermitian operator in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can always be written as a difference of two positive operators P and Q:

$$W = Q - P, \tag{61}$$

and, as is well known, most of known EWs have this form with Q being separable (very often $Q \propto \mathbb{I}_1 \otimes \mathbb{I}_2$, but following [31] one can look for more general form of Q) and P being entangled (for example maximally entangled pure state). Let W defined in (61) be an EW detecting an NPT (and hence entangled) state P. Is this W indecomposable? One may try to look for the states detectable by W in the following form

$$\rho_\alpha = (1-\alpha)P + \alpha\sigma_{\mathrm{sep}}, \tag{62}$$

where σ_α is a separable state. Now, mixing an NPT state P with σ_{sep} may result in a PPT state. Hence, if ρ_α becomes PPT for some $\alpha > 0$, and it is still detected by W, then W is necessarily an indecomposable EW.

Conversely, given a PPTES state ρ one may try to construct the corresponding (indecomposable) EW detecting ρ. This problem is in general very complex since it is extremely difficult checking weather W satisfies $\mathrm{Tr}(W\sigma_{\mathrm{sep}}) \geqslant 0$ for *all* separable σ_{sep}. One example of such construction is provided via unextendible product bases by Terhal in [7].

It is clear, that the method presented provides new classes of indecomposable (and atomic) linear positive maps (for recent analysis of atomic maps see [30]). In particular a positive map corresponding to $W_{\lambda,\mu}$ defined in (41) provides a considerable generalization of the Choi map. On may try looking for other well known positive indecomposable maps and perform 'deformation' within the class of indecomposable maps. Any new examples of such maps provide an important tool for studies of quantum entanglement.

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